Computers and Mathematics with Applications 64 (2012) 1170–1178

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Stabilization of uncertain fuzzy control systems via a new descriptor system approach

Kuo-Jung Lin *

Department of Electrical Engineering, Fortune Institute of Technology, Kaohsiung City 83160, Taiwan, ROC

ARTICLE INFO

Keywords:
Descriptor system
H^{∞} control performance
Linear matrix inequality (LMI)
T–S fuzzy model
Uncertain nonlinear systems

ABSTRACT

This paper addresses stabilizing a class of fuzzy control systems with a guaranteed H^{∞} control performance via a new descriptor system approach. Based on the sector nonlinearity concept of Tanaka and Wang (2001) [1], the uncertain nonlinear system can be exactly represented by T–S fuzzy models. Then, we propose using the composite state and output feedback (CSAOF) fuzzy control for the control design. A new descriptor fuzzy system will be represented in this paper. Based on the Lyapunov stability theorem and the linear matrix inequality (LMI) tool, we solve the controller gain matrices, some positive constants and some common positive-definite matrices. Then, we derive two sufficient conditions to stabilize the uncertain fuzzy control systems with guaranteed H^{∞} control performance. Moreover, the developed H^{∞} criterion guarantees that the influence of external disturbance is as small as possible. A practical system is given to illustrate the validity of the proposed scheme.

1. Introduction

During the past two decades, much literature have been presented to deal with fuzzy control applications (see [1–15] and the references therein). A T–S fuzzy model can be used to approximate a nonlinear model. However, using the sector nonlinearity concept [1], the nonlinear system can exactly represent the T–S fuzzy model. In this type of fuzzy model, every local subsystem can be represented as local linear subsystem. The overall system is approximate or exact to a nonlinear system. The theory of linear matrix inequalities (LMIs) [16] has been widely used as a tool in order to solve the stability analysis and control design of the fuzzy control system.

Recently, stability analysis and control design of descriptor systems have been extensively studied [17,11,18–20]. However, some papers propose a descriptor system approach to design uncertain linear systems [17] and fuzzy systems [11]. In [17,11], the authors all use state feedback control designed to stabilize systems. During the last few decades, robust control designs have been extensively studied [17,20,12,21–24]. Furthermore, many researchers focus on the H^{∞} control design problem [18]. The objective of H^{∞} control design is to inhibit the influence of external disturbances.

The output feedback and dynamic output feedback control design for control systems have been found in the literature [13,14,25]. The output feedback or dynamic output feedback control problem turns out to be much more difficult to solve compared to state feedback control problems. An important reason for using dynamic output feedback control design is that many problems involving synthesizing dynamic controllers can be formulated as output feedback control problems involving augmented plants. In [13], by some assumptions, the dynamic output feedback control problem becomes a state feedback control problem. Recently, the output feedback control design has been applied to fuzzy control systems, such as

* Tel.: +886 7 7889888; fax: +886 7 7889777.
E-mail addresses: kuo5jung@seed.net.tw, t1216@center.fotech.edu.tw.

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doi:10.1016/j.camwa.2012.03.059

in [13,14]. In [14,25], the static output feedback stabilizing problem is addressed using an iterative linear matrix inequality (ILMI) technique.

In this paper, we propose using the composite state and output feedback (CSAOF) fuzzy control for the control design. The contribution is that we propose a new descriptor system approach; we can easily solve the state and output feedback fuzzy controller gain matrices via LMI technology but not ILMI. An inverted pendulum controlled by a DC motor as the uncertain nonlinear system is given to examine the proposed method.

Notation: $R^{n\times m}$ is the set of all $n \times m$ real matrices. I is an identity matrix with approximate dimensions and I_{r_6} denotes an identity matrix with dimensions $r_6 \times r_6$. The notation $\bar{X} < \bar{Y}$ means that $\bar{X} - \bar{Y} < 0$. In other words, $\bar{X} - \bar{Y}$ is a negative definite matrix. The value 0 denotes a zero matrix with approximate dimensions. 0_{r_6} denotes a zero matrix with the dimensions $r_6 \times r_6$, and $\lambda_{\max}(\bar{M})$ denotes the maximum eigenvalue of $\bar{M}$. $\sigma_{\min}(\bar{M})$ denotes the smallest singular value of $\bar{M}$.

2. Problem formulation

Consider the following uncertain nonlinear system:

$$\dot{x}(t) = \bar{f}(x(t), u(t)) + Ew(t) \tag{1a}$$

$$y(t) = \bar{g}(x(t), u(t)) \tag{1b}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in R^{n\times 1}$ denote the state vectors, $\bar{f}(x(t), u(t))$ and $\bar{g}(x(t), u(t))$ are uncertain nonlinear functions, $u(t) \in R^{r_3}$ is the control input, $w(t) \in R^{r_2}$ represents the unknown but bounded disturbances (or noises) with an upper bound of $\|w(t)\| \le d$, $E \in R^{n\times r_2}$ is the constant matrix, and $y(t) \in R^{r_1\times 1}$ is the output signal. Based on the sector nonlinearity concept [1], the uncertain nonlinear system (1) can be exactly represented as the following Takagi–Sugeno (T–S) fuzzy model (2):

Plant Rule i:
IF $q_1(t)$ is M_{i1} and ...and $q_p(t)$ is M_{ip}
THEN

$$\dot{x}(t) = (A_i + \Delta A_i)x(t) + (B_i + \Delta B_i)u(t) + Ew(t) \tag{2a}$$

$$y(t) = (C_i + \Delta C_i)x(t) + (D_i + \Delta D_i)u(t) \tag{2b}$$

where $i = 1, 2, \ldots, r, A_i \in R^{n\times n},\ B_i \in R^{n\times r_3}, y(t) \in R^{r_1\times 1}, C_i \in R^{r_1\times n}$ and $D_i \in R^{r_1\times r_3}$, M_{ij} is the fuzzy term-set, r is the number of IF-THEN rules, $q_1(t) \sim q_p(t)$ are the premise variables, and ΔA_i, ΔB_i, ΔC_i and ΔD_i are uncertain matrices. In this paper, we assume $\Delta B_i = B_i D$ and $\Delta D_i = D_i D$, where $D \in R^{r_3\times r_3}$. D is a constant square matrix. Then, the overall uncertain fuzzy system is inferred as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i\{(A_i + \Delta A_i)x(t) + B_i(I_{r_3} + D)u(t) + Ew(t)\} \tag{3a}$$

$$y(t) = \sum_{i=1}^{r} h_i\{(C_i + \Delta C_i)x(t) + D_i(I_{r_3} + D)u(t)\} \tag{3b}$$

where $h_i \equiv h_i(q(t)) = \frac{\mu_i(q(t))}{\sum_{i=1}^{r}\mu_i(q(t))}$, $q(t) = [q_1(t) \quad q_2(t) \quad \cdots \quad q_p(t)]$, $\mu_i(q(t)) = \prod_{j=1}^{p} M_{ij}(q_j(t))$, $\begin{cases}\sum_{i=1}^{r}\mu_i(q(t)) > 0 \\ \mu_i(q(t)) \ge 0\end{cases}$, $i = 1, 2, \ldots, r$.

For all t, $M_{ij}(q(t))$ is the membership grade of $q_j(t)$ in M_{ij}. From (3) we have the following:

$$\begin{cases}\displaystyle\sum_{i=1}^{r} h_i(q(t)) = 1 \\ h_i(q(t)) \ge 0\end{cases}, \quad i = 1, 2, \ldots, r. \tag{4}$$

In the following, we propose the CSAOF fuzzy control for the control design. The ith rule of the CSAOF fuzzy controller is of the following form:

Control Rule i:
IF $q_1(t)$ is M_{i1} and ...and $q_p(t)$ is M_{ip}
THEN

$$u(t) = (I_{r_3} + D)^{-1}[K_i x(t) + G_i y(t)], \quad i = 1, 2, \ldots, r \tag{5}$$

where $K_i \in R^{r_3\times n}$ are state feedback gain matrices and $G_i \in R^{r_3\times r_1}$ are output feedback gain matrices. The overall CSAOF fuzzy controller can be represented in the following form:

$$u(t) = \sum_{i=1}^{r} h_i(I_{r_3} + D)^{-1}[K_i x(t) + G_i y(t)]. \tag{6}$$

To represent the uncertain fuzzy system into the descriptor fuzzy system, the output Eq. (3b) is converted into (7), as follows:

$$0_{r_1} \cdot \dot{y}(t) = -y(t) + \sum_{i=1}^{r} h_i\{(C_i + \Delta C_i)x(t) + D_i(I_{r_3} + D)u(t)\} \tag{7}$$

where $0_{r_1} \in R^{r_1 \times r_1}$ denotes the matrix whose elements all are zero. From (6), (7) and (3a), we obtain the representation of the following uncertain descriptor system:

$$\Gamma \dot{X}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j[(\bar{A}_i + \bar{B}_i\bar{K}_j + \Delta\bar{A}_i)X(t) + \bar{E}w(t)] \tag{8}$$

where $X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, $\Gamma = \begin{bmatrix} I_n & 0 \\ 0 & 0_{r_1} \end{bmatrix}$, $\bar{A}_i = \begin{bmatrix} A_i & 0 \\ C_i & -I_{r_1} \end{bmatrix}$, $\bar{B}_i = \begin{bmatrix} B_i & B_i \\ D_i & D_i \end{bmatrix}$, $\bar{K}_j = \begin{bmatrix} K_j & 0 \\ 0 & G_j \end{bmatrix}$, $\Delta\bar{A}_i = \begin{bmatrix} \Delta A_i & 0 \\ \Delta C_i & 0_{r_1} \end{bmatrix}$, and $\bar{E} = \begin{bmatrix} E \\ 0 \end{bmatrix}$. If $\bar{E} = 0$, we get the following properties: If $\det(S\Gamma - \bar{A}_i - \bar{B}_i\bar{K}_j - \Delta\bar{A}_i)$ is not identically zero, then the pair $(\Gamma,\ \bar{A}_i + \bar{B}_i\bar{K}_j + \Delta\bar{A}_i)$ is regular. If all the finite poles $(\Gamma,\ \bar{A}_i + \bar{B}_i\bar{K}_j + \Delta\bar{A}_i)$ lie in the negative plane of S, then $(\Gamma,\ \bar{A}_i + \bar{B}_i\bar{K}_j + \Delta\bar{A}_i)$ is said to be stable. If $\deg(\det(S\Gamma - \bar{A}_i - \bar{B}_i\bar{K}_j - \Delta\bar{A}_i)) = \mathrm{rank}(\Gamma)$, then $(\Gamma,\ \bar{A}_i + \bar{B}_i\bar{K}_j + \Delta\bar{A}_i)$ is called impulse free, where $\mathrm{rank}(\Gamma) \le (n + r_1)$. $(\Gamma,\ \bar{A}_i + \bar{B}_i\bar{K}_j + \Delta\bar{A}_i)$ is admissible, if it is regular, stable and impulse free.

The objective of the following section is to determine how to design the fuzzy controller (6) such that the system (8) is globally asymptotically stable.

3. Robust fuzzy control design via descriptor system approach

In this section, we design the CSAOF fuzzy controller (6) to stabilize the uncertain descriptor system (8). In the following, we consider a two-type representation of the uncertainty, $\Delta\bar{A}_i$. First, we assume $\Delta\bar{A}_i = M_i F N_i$ and $F^T F \le I$. To prove the main theorem, we define the following matrices: $M_i = \begin{bmatrix} M_{i1} \\ M_{i2} \end{bmatrix}$, $N_i = \begin{bmatrix} N_{i1} & N_{i2} \end{bmatrix}$, $P = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix}$,where $P > 0$, $P_{11} = P_{11}^T > 0$ and $P_{22} = P_{22}^T > 0$. $Q = P^{-1}$, $Q = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix}$, where $Q > 0$, $Q_{11} = P_{11}^{-1}$ and $Q_{22} = P_{22}^{-1}$. $\bar{K}_j = \bar{H}_j P$, where $\bar{H}_j = \begin{bmatrix} H_{1j} & 0 \\ 0 & H_{2j} \end{bmatrix}$, $K_j = H_{1j}P_{11}$and $G_j = H_{2j}P_{22}$. $R_1 = \begin{bmatrix} R_{111} & R_{112} \\ R_{112}^T & R_{122} \end{bmatrix}$, where $R_1 > 0$, $R_{111} = R_{111}^T > 0$ and $R_{122} = R_{122}^T > 0$.

Furthermore, we recall the following lemmas:

Lemma 1 ([15]). *The following is known for any constant ζ and any matrices Φ and ξ:*

$$\Phi^T\xi + \xi^T\Phi \le \zeta\, \Phi^T\Phi + \zeta^{-1}\xi^T\xi. \tag{9}$$

Lemma 2 ([16] *Schur Complement Formula*). *If $\bar{Q} = \bar{Q}^T$ and $\bar{R} = \bar{R}^T$, then the LMI $\begin{bmatrix} \bar{Q} & \bar{S} \\ \bar{S}^T & \bar{R} \end{bmatrix} > 0$ is equivalent to $\bar{Q} > 0$ and $\bar{R} - \bar{S}^T\bar{Q}^{-1}\bar{S} > 0$ or $\bar{R} > 0$ and $\bar{Q} - \bar{S}\bar{R}^{-1}\bar{S}^T > 0$.*

Now, we consider the following H^∞ control performance:

$$\int_0^{t_f} X^T(t)PR_1PX(t)dt < X^T(0)\Gamma PX(0) + \gamma_1^{-1}\lambda_{\max}(\bar{E}^T\bar{E})\int_0^{t_f} w^T(t)w(t)dt \tag{10}$$

where t_f is the terminal control time. Then, a sufficient condition for stabilizing the descriptor system (8) under the CSAOF fuzzy control (6) is established in the following theorems:

Theorem 1. *The uncertain descriptor system (8) is regular, impulse free and global asymptotically stable, if there exist positively defined symmetry matrices Q_{11}, Q_{22}, R_{111} and R_{122}, constant matrices R_{112}, H_{1j}, and H_{2j}, and positive constants γ_1 and γ_2 that satisfy the following LMIs:*

$$\begin{bmatrix} \Xi & \Theta & Q_{11}N_{i1}^T \\ \Theta^T & \Psi & Q_{22}N_{i2}^T \\ N_{i1}Q_{11} & N_{i2}Q_{22} & -\gamma_2 I \end{bmatrix} < 0, \quad i, j = 1, 2, \ldots, r. \tag{11a}$$

$$\begin{bmatrix} R_{111} & R_{112} \\ R_{112}^T & R_{122} \end{bmatrix} > 0 \tag{11b}$$

where

$$\Xi = R_{111} + A_iQ_{11} + Q_{11}A_i^T + H_{1j}^TB_i^T + B_iH_{1j} + \gamma_1 I_n + \gamma_2 M_{i1}M_{i1}^T,$$

$$\Theta = R_{112} + Q_{11}C_i^T + H_{1j}^TD_i^T + B_iH_{2j} + \gamma_2 M_{i1}M_{i2}^T$$

$$\Psi = R_{122} - 2Q_{22} + H_{2j}^TD_i^T + D_iH_{2j} + \gamma_1 I_{r_1} + \gamma_2 M_{i2}M_{i2}^T.$$

Proof. Define the Lyapunov function as $V(t) = X^T(t)\Gamma PX(t)$

Then

$$\begin{aligned}\dot{V}(t) &= \dot{X}^T(t)\Gamma PX(t) + X^T(t)P\Gamma\dot{X}(t)\\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_ih_j\{[X^T(\bar{A}_i + \bar{B}_i\bar{K}_j)^T + X^T(M_iFN_i)^T + w^T\bar{E}^T]PX\\ &\quad + XP[(\bar{A}_i + \bar{B}_i\bar{K}_j)X + M_iFN_iX + \bar{E}w]\}\\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_ih_j\{[X^T(\bar{A}_i + \bar{B}_i\bar{K}_j)^TP + P(\bar{A}_i + \bar{B}_i\bar{K}_j) + (M_iFN_i)^TP\\ &\quad + P(M_iFN_i)]X + w^T\bar{E}^TPX + X^TP\bar{E}w\}.\end{aligned}$$

Applying Lemma 1, the equality above becomes the following:

$$\begin{aligned}\dot{V} &\le \sum_{i=1}^{r}\sum_{j=1}^{r} h_ih_j\{[X^TP[Q(\bar{A}_i + \bar{B}_i\bar{K}_j)^T + (\bar{A}_i + \bar{B}_i\bar{K}_j)Q\\ &\quad + QN_i^TF^TM_i^T + M_iFN_iQ]PX + X^TP\gamma_1PX + w^T\bar{E}^T\gamma_1^{-1}\bar{E}w]\}\\ &\le \sum_{i=1}^{r}\sum_{j=1}^{r} h_ih_j\{[X^TP[Q\bar{A}_i^T + \bar{A}_iQ + \bar{H}_j^T\bar{B}_i^T + \bar{B}_i\bar{H}_j + QN_i^T\gamma_2^{-1}NQ + \gamma_2M_iM_i^T + \gamma_1I]PX\\ &\quad + \gamma_1^{-1}\lambda_{\max}(\bar{E}^T\bar{E})w^Tw]\}.\end{aligned}$$

If

$$Q\bar{A}_i^T + \bar{A}_iQ + \bar{H}_j^T\bar{B}_i^T + \bar{B}_i\bar{H}_j + QN_i^T\gamma_2^{-1}NQ + \gamma_2M_iM_i^T + \gamma_1I < -R_1. \tag{12}$$

Then

$$\dot{V} < X^TP(-R_1)PX + \gamma_1^{-1}\lambda_{\max}(\bar{E}^T\bar{E})w^Tw. \tag{13}$$

Integrating (13) from $t = 0$ to $t = t_f$ yields the following:

$$V(t_f) - V(0) < -\int_0^{t_f} X^TPR_1PXdt + \gamma_1^{-1}\lambda_{\max}(\bar{E}^T\bar{E})\int_0^{t_f} w^Twdt. \tag{14}$$

From (14), we get (10). Furthermore, by using Lemma 2 to (12) implies (11) holds. This completes the proof.

Second, we assume that the uncertainty $\Delta\bar{A}_i = \sum_{k=1}^{m}\alpha_{ik}\bar{A}_{ik}$, where α_{ik} denotes the real constant. Define $R_2 = \begin{bmatrix} R_{211} & R_{212}\\ R_{212}^T & R_{222}\end{bmatrix}$, where $R_2 > 0$, $R_{211} = R_{211}^T > 0$ and $R_{222} = R_{222}^T > 0$. Now, we consider the following H^∞ control performance:

$$\gamma_4\int_0^{t_f} X^T(t)X(t)dt < X^T(0)\Gamma PX(0) + \gamma_3^{-1}\lambda_{\max}(\bar{E}^T\bar{E})\int_0^{t_f} w^T(t)w(t)dt \tag{15}$$

where t_f is terminal control time. Then, a sufficient condition to stabilize the descriptor system (8) under the CSAOF fuzzy control (6) is established in Theorem 2, and the allowable perturbation bound, α_{ik} is calculated. □

Theorem 2. *The uncertain descriptor system (8) is regular, impulse free, and global asymptotically stable, if there exist positively defined symmetry matrices Q_{11}, Q_{22}, R_{211}, and R_{222}, constant matrices R_{212}, H_{1j}, and H_{2j}, and positive constants γ_3 and γ_4 that satisfy the following LMIs:*

$$\begin{bmatrix}\bar{\Xi} & \bar{\Theta}\\ \bar{\Theta}^T & \bar{\Psi}\end{bmatrix} < 0, \quad i, j = 1, 2, \ldots, r. \tag{16a}$$

$$\begin{bmatrix} R_{211} - \gamma_4I & R_{212}\\ R_{212}^T & R_{222} - \gamma_4I\end{bmatrix} > 0 \tag{16b}$$

and the allowable perturbation bound, α_{ik} can be calculated as follows

$$\sum_{k=1}^{m}|\alpha_{ik}|^2 \le \frac{\sigma_{\min}^2[P(R_2 - \gamma_4I)P]}{\sum_{k=1}^{m}\|\bar{A}_{ik}^TP + P\bar{A}_{ik}\|^2}, \quad i = 1, 2, \ldots, r \tag{17}$$

where

$$\bar{\Xi} = R_{211} + A_iQ_{11} + Q_{11}A_i^T + B_iH_{1j} + H_{1j}^TB_i^T + \gamma_3 I,$$
$$\bar{\Theta} = R_{212} + Q_{11}C_i^T + H_{1j}^TD_i^T + B_iH_{2j}$$
$$\bar{\Psi} = R_{222} - 2Q_{22} + H_{2j}^TD_i^T + D_iH_{2j} + \gamma_3 I_{r_1}.$$

Proof. From the proof of Theorem 1, we have

$$\begin{aligned}\dot{V} &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_ih_j\left\{\left[X^T\left[(\bar{A}_i+\bar{B}_i\bar{K}_j)^TP + P(\bar{A}_i+\bar{B}_i\bar{K}_j) + \sum_{k=1}^{m}\alpha_{ik}(\bar{A}_{ik}^TP + P\bar{A}_{ik})\right]X + w^T\bar{E}^TPX + X^TP\bar{E}w\right]\right\}\\
&\le \sum_{i=1}^{r}\sum_{j=1}^{r} h_ih_j\left\{\left[X^TP\left[Q(\bar{A}_i+\bar{B}_i\bar{K}_j)^T + (\bar{A}_i+\bar{B}_i\bar{K}_j)Q + \sum_{k=1}^{m}\alpha_{ik}(Q\bar{A}_{ik}^T + \bar{A}_{ik}Q)\right]PX + X^TP\gamma_3PX + w^T\bar{E}^T\gamma_3^{-1}\bar{E}w\right]\right\}\\
&\le \sum_{i=1}^{r}\sum_{j=1}^{r} h_ih_j\left\{\left[X^TP\left[Q\bar{A}_i^T + \bar{A}_iQ + \bar{H}_j^T\bar{B}_i^T + \bar{B}_i\bar{H}_j + \sum_{k=1}^{m}\alpha_{ik}(Q\bar{A}_{ik}^T + \bar{A}_{ik}Q) + \gamma_3 I\right]PX + \gamma_3^{-1}\lambda_{\max}(\bar{E}^T\bar{E})w^Tw\right]\right\}.\end{aligned}$$

If

$$Q\bar{A}_i^T + \bar{A}_iQ + \bar{H}_j^T\bar{B}_i^T + \bar{B}_i\bar{H}_j + \gamma_3 I < -R_2. \tag{18}$$

Then

$$\dot{V} < \sum_{i=1}^{r}\sum_{j=1}^{r} h_ih_j\left\{X^TP\left[-R_2 + \sum_{k=1}^{m}\alpha_{ik}(Q\bar{A}_{ik}^T + \bar{A}_{ik}Q)\right]PX + \gamma_3^{-1}\lambda_{\max}(\bar{E}^T\bar{E})w^Tw\right\} \tag{19}$$

$$= \sum_{i=1}^{r}\sum_{j=1}^{r} h_ih_j\left\{-X^TP\gamma_4PX + X^TP\left[-(R_2-\gamma_4 I) + \sum_{k=1}^{m}\alpha_{ik}(Q\bar{A}_{ik}^T + \bar{A}_{ik}Q)\right]PX + \gamma_3^{-1}\lambda_{\max}(\bar{E}^T\bar{E})w^Tw\right\}. \tag{20}$$

If $R_2 > \gamma_4 I$ and

$$X^TP\left[-(R_2-\gamma_4 I) + \sum_{k=1}^{m}\alpha_{ik}(Q\bar{A}_{ik}^T + \bar{A}_{ik}Q)\right]PX \le 0. \tag{21}$$

Then, (20) becomes the following:

$$\dot{V} < -X^TP\gamma_4PX + \gamma_3^{-1}\lambda_{\max}(\bar{E}^T\bar{E})w^Tw. \tag{22}$$

Integrating (22) from $t = 0$ to $t = t_f$ yields the following:

$$V(t_f) - V(0) < -\int_0^{t_f} X^TP\gamma_4PXdt + \gamma_3^{-1}\lambda_{\max}(\bar{E}^T\bar{E})\int_0^{t_f} w^Twdt. \tag{23}$$

From (23), we get (15). Furthermore, by using Lemma 2 to (18) implies that (16a) holds. Furthermore, because $R_2 > \gamma_4 I$, then (16b) holds. From (21), we have the following:

$$\begin{aligned}& X^T\sum_{k=1}^{m}\alpha_{ik}(\bar{A}_{ik}^TP + P\bar{A}_{ik})X \le X^TP(R_2-\gamma_4 I)PX\\
&\Rightarrow \left|X^T\sum_{k=1}^{m}\alpha_{ik}(\bar{A}_{ik}^TP + P\bar{A}_{ik})X\right| \le \|X\|^2\sum_{k=1}^{m}|\alpha_{ik}|\,\|\bar{A}_{ik}^TP + P\bar{A}_{ik}\|\\
&\quad \le \sigma_{\min}[P(R_2-\gamma_4 I)P]X^TX\\
&\Rightarrow \begin{bmatrix}\alpha_{i1} & \alpha_{i2} & \cdots & \alpha_{im}\end{bmatrix}\cdot\begin{bmatrix}\|\bar{A}_{i1}^TP + P\bar{A}_{i1}\|\\ \|\bar{A}_{i2}^TP + P\bar{A}_{i2}\|\\ \vdots\\ \|\bar{A}_{im}^TP + P\bar{A}_{im}\|\end{bmatrix} < \sigma_{\min}[P(R_2-\gamma_4 I)P]\end{aligned}$$

$$\Rightarrow \sum_{k=1}^{m}|\alpha_{ik}|^2\cdot\sum_{k=1}^{m}\|\bar{A}_{ik}^TP + P\bar{A}_{ik}\|^2 \le \sigma_{\min}^2[P(R_2-\gamma_4 I)P]. \tag{24}$$

From (24), we known that (17) holds. This completes the proof. □

Remark. If $H_{2i} = 0, i = 1, 2, \ldots, r$, the behavior of the fuzzy controller (6) is state feedback fuzzy control. If $H_{1i} = 0, i = 1, 2, \ldots, r$, the behavior of the fuzzy controller (6) is output feedback fuzzy control.

4. Numerical example

We consider an inverted pendulum controlled by a DC motor [12] as the uncertain nonlinear system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ 9.8 \sin x_1 + x_3 \\ -10x_2 - 10x_3 \end{bmatrix} + \Delta A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} (I + D)u(t) + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} w(t) \tag{25a}$$

$$y(t) = \left(\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} + \Delta C\right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 2(I + D)u(t) \tag{25b}$$

where x_1 is the angle of the pendulum, $x_2 = \dot{x}_1$, and x_3 is the current of the dc motor and $w(t)$ represents external disturbances, $w(t) = 2 \sin t$. We assume that $x_1(t) \in [-0.5,\ 0.5]$. The nonlinear term is $\sin x_1$. Using the sector nonlinearity concept [1], the nonlinear term can be represented as follows:

$$\sin x_1(t) = M_1(x_1(t)) \cdot 1 \cdot x_1(t) + M_2(x_1(t)) \cdot \frac{\sin 0.5}{0.5} \cdot x_1(t)$$

where $M_1(x_1(t)),\ M_2(x_1(t)) \in [0,\ 1]$ and $M_1(x_1(t)) + M_2(x_1(t)) = 1$. By solving the above equation, we get the following membership functions:

$$h_1 = M_1(x_1(t)) = \begin{cases} \dfrac{0.5 \cdot \sin x_1(t) - x_1(t) \cdot \sin 0.5}{x_1(t) \cdot (0.5 - \sin 0.5)}, & x_1(t) \neq 0 \\ 1, & x_1(t) = 0 \end{cases} \tag{26a}$$

$$h_2 = M_2(x_1(t)) = \begin{cases} \dfrac{0.5(x_1(t) - \sin x_1(t))}{x_1(t) \cdot (0.5 - \sin 0.5)}, & x_1(t) \neq 0 \\ 0, & x_1(t) = 0. \end{cases} \tag{26b}$$

From (26), the system given in (25) can be represented by the following T–S fuzzy model.
Plant Rule 1:
IF $x_1(t)$ is $M_1(x_1(t))$;
THEN

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 & 0 \\ 9.8 & 0 & 1 \\ 0 & -10 & -10 \end{bmatrix} + \Delta A \right) \begin{bmatrix} x_1(t) \\ x_2(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} (I + D)u(t) + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} w(t)$$

$$y(t) = \left(\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} + \Delta C \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 2(I + D)u(t).$$

Plant Rule 2:
IF $x_1(t)$ is $M_2(x_1(t))$;
THEN

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 & 0 \\ \dfrac{9.8 \sin 0.5}{0.5} & 0 & 1 \\ 0 & -10 & -10 \end{bmatrix} + \Delta A \right) \begin{bmatrix} x_1(t) \\ x_2(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} (I + D)u(t) + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} w(t)$$

$$y(t) = \left(\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} + \Delta C \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 2(I + D)u(t).$$

If $\Delta A = \begin{bmatrix} 0.01 & 0 & 0.01 \\ 0.02 & 0 & 0.02 \\ 0.01 & 0 & 0.01 \end{bmatrix}$, $\Delta C = \begin{bmatrix} 0.05 & 0 & 0.05 \end{bmatrix}$, $D = 3$, then we have $\Delta \bar{A}_i = M_i F N_i$, $M_i = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.1 \\ 0.5 \end{bmatrix}$, $F = 1$, $N_i = \begin{bmatrix} 0.1 & 0 & 0.1 & 0 \end{bmatrix}$, $M_{i1} = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.1 \end{bmatrix}$, $M_{i2} = 0.5$, $N_{i1} = \begin{bmatrix} 0.1 & 0 & 0.1 \end{bmatrix}$ and $N_{i2} = 0, i = 1, 2$.

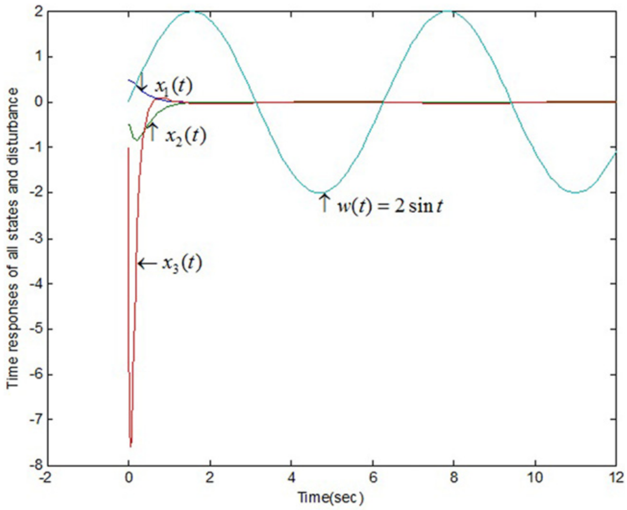

Fig. 1. Simulation results of applying the proposed control (27) to the uncertain nonlinear system (25).

By way of the LMI procedure (11), we solve the following:

$$R_{111} = 10^4 \times \begin{bmatrix} 0.1885 & -0.5514 & -0.1182 \\ -0.5514 & 1.7789 & 0.1976 \\ -0.1182 & 0.1976 & 1.9046 \end{bmatrix}, \qquad R_{112} = 10^4 \times \begin{bmatrix} 0.6805 \\ -1.1728 \\ -1.0913 \end{bmatrix},$$

$$R_{122} = 7.2936 \times 10^4, \qquad Q_{11} = 10^4 \times \begin{bmatrix} 0.0538 & -0.1700 & -0.0933 \\ -0.1700 & 0.5840 & -0.0927 \\ -0.0933 & -0.0927 & 3.7190 \end{bmatrix},$$

$$Q_{22} = 3.1678 \times 10^4, \qquad H_{11} = H_{12} = 10^4 \times \begin{bmatrix} -0.2391 & 0.1865 & 3.4297 \end{bmatrix},$$

$$H_{21} = H_{22} = -9.0655 \times 10^3, \qquad \gamma_1 = 19.2377, \qquad \gamma_2 = 4.7975 \times 10^3,$$

$$K_1 = K_2 = \begin{bmatrix} -163.5426 & -47.9776 & -4.3765 \end{bmatrix} \quad \text{and} \quad G_1 = G_2 = -0.2862.$$

The CSAOF fuzzy controller is the following:

$$u(t) = -40.8856x_1(t) - 11.9944x_2(t) - 1.0941x_3(t) - 0.0716y(t). \tag{27}$$

To examine the validity, we apply the proposed control (27) to the uncertain nonlinear system (25). Simulation results of the states under the initial values $x_1(0) = 0.5$, $x_2(0) = -0.5$, and $x_3(0) = -1$ are shown in Fig. 1. Obviously, the resulting controller stabilizes the uncertain nonlinear system (25).

If we assume the uncertainty, $\Delta\bar{A}_i = \sum_{k=1}^{2} \alpha_{ik}\bar{A}_{ik}$, where $\bar{A}_{i1} = \begin{bmatrix} 0 & 0 & 0.0001 & 0 \\ 0 & 0.0001 & 0 & 0 \\ 0 & 0 & 0.0001 & 0 \\ 0.0001 & 0 & 0 & 0 \end{bmatrix}$, $\bar{A}_{i2} = \begin{bmatrix} 0 & 0 & 0.0001 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0001 & 0 \\ 0 & 0.0001 & 0 & 0 \end{bmatrix}$, $i = 1, 2$, then by way of the LMI procedure (16), we solve $R_{211} = \begin{bmatrix} 66.5178 & -198.9308 & -39.6823 \\ -198.9308 & 657.7386 & 71.9395 \\ -39.6823 & 71.9395 & 697.0657 \end{bmatrix}$, $R_{212} = \begin{bmatrix} 191.3064 \\ -325.1271 \\ -299.0487 \end{bmatrix}$, $R_{222} = 1921.1$, $Q_{11} = 10^3 \times \begin{bmatrix} 0.0186 & -0.0607 & -0.0169 \\ -0.0607 & 0.2126 & -0.0618 \\ -0.0169 & -0.0618 & 1.1293 \end{bmatrix}$, $Q_{22} = 888.3088$, $H_{11} = H_{12} = \begin{bmatrix} -66.141 & 45.2919 & 999.7635 \end{bmatrix}$, $\gamma_3 = 0.8207$, $\gamma_4 = 0.8150$, $K_1 = K_2 = \begin{bmatrix} -115.9245 & -33.6559 & -2.6892 \end{bmatrix}$,and $G_1 = G_2 = -0.2968$. From (17), we solve the allowable perturbation bounds, which are $\alpha_{i1}^2 + \alpha_{i2}^2 \leq 0.0304$, $i = 1, 2$.

The CSAOF fuzzy controller is the following:

$$u(t) = -28.9811x_1(t) - 8.4140x_2(t) - 0.6723x_3(t) - 0.0742y(t). \tag{28}$$

To examine the validity, we apply the proposed control (28) to the uncertain nonlinear system (25). Simulation results of the states under the initial values $x_1(0) = 0.5$, $x_2(0) = -0.5$, and $x_3(0) = -1$ are shown in Fig. 2. Obviously, the resulting controller stabilizes the uncertain nonlinear system (25).

5. Conclusions

A new descriptor system approach to stabilizing a class of fuzzy control systems with guaranteed H^∞ control performance, and the CSAOF fuzzy control for the control design are presented. Based on the Lyapunov stability theorem

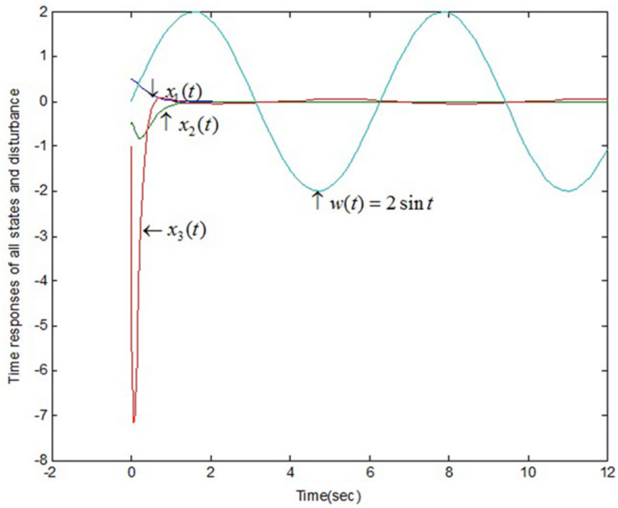

Fig. 2. Simulation results of applying the proposed control (28) to the uncertain nonlinear system (25).

and the LMIs, we determine all the controller gain matrices. The proposed CSAOF fuzzy controller will stabilize uncertain nonlinear systems. The effect of external disturbance on control performance is attenuated to a minimum level. The advantage we show is that the proposed CSAOF fuzzy control designed to stabilize the uncertain nonlinear systems, and all the controller gain matrices can be solved by LMI technology. We use a DC motor-driven inverted-pendulum system to illustrate the modeling and design procedure. The simulation results demonstrate that the proposed scheme is feasible and satisfactory.

Acknowledgments

This work was partly supported by the National Science Council, Taiwan, the Republic of China, under Grants NSC 100-2221-E-268-003.

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